When Global Games meet Micro-founded Sovereign Debt Models *

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September, 2012

Abstract

Modeling the sovereign debt, and predicting the associated risk of crises may have interesting implications in terms of public policy analysis. Nonetheless until now we had to make a choice between two categories of sovereign debt models, with opposed advantages and drawbacks. On the one hand, micro-founded sovereign debt models have been developed following CALVO(1988). They yield a developed market microstructure, but we must deal with the multiplicity of equilibria, which prevents us from anticipating risks of crises. On the other hand, the global games literature has enabled the study of regime change models - amongst which sovereign debt crises - particularly since its development by MORRIS & SHIN(2003,2004A,2004B). Here, the model has been often poorly treated, but the original multiplicity of equilibria that arises in classical coordination games has been removed.

The aim of this paper is to try and reconciliate these two literatures by developing global games models with a more developed market microstructure, in order to keep the best part of these two kinds of literature. The belief that drove us in the construction of our models is that the return on the government bonds plays a central role in sovereign debt issues. The first model we focused on represents a stylized secondary market, where the different agents observe an exogenous return, assumed to approximately reflect the State of the economy. One interesting consequence is that, if the agents don't trust anymore that the interest rate reflects correctly the true value of the fundamentals, the better way for the State to avoid default is to let the value of the interest rate increase. In a second model we add an asset pricing market to the game, in order to take into account the primary market too, and to endogenize the formation of the interest rate. Here, a similar consequence States that if the agents become very uncertain about the future, the higher the interest rate, the lower the risk of default.

Nevertheless, this last model calls for some important restrictions of the game, to avoid the multiplicity of equilibria generated by the first-stage asset market. Future research could focus on this point, in order to loosen these restrictions.

*I would like to thank Edouard Challe for his support along the preparation and redaction of this paper

1 Introduction

In a seminal paper, Calvo(1988) builds a first micro-founded model of sovereign debt crisis. In this model, a benevolent government tries to maximize the utility of a representative individual, who is forced to buy sovereign bonds. This government has thus the ability to repudiate part of his debt if it appears to increase the utility of this individual, compared to an equivalent distorsionary taxation. Under given conditions, this model exhibits two perfect-foresight equilibria, which are Pareto-ranked. In a good equilibrium there is no default on the debt: the State is able to fully reimburse its creditors, while in the bad equilibrium, the State is forced to declare partial default on its debt.

One drawback, of this model is that it does not provide any insight in the ability to predict which equilibrium will occur. The only proposed solution is to set a maximum return on the auctioned bond, which is not completely satisfactory. Moreover, letting the multiplicity of equilibria in the model may appear counter-intuitive for different reasons. The first one is that the equilibrium selection calls for shifts in beliefs of the agents, whose cause can't be explained in this framework; the second one, that it doesn't enable to generate any link between the fundamentals underlying the economy, and the economical outcomes of the model. This is then important to try and determine a model, where we could end up this multiplicity.

The intuitive way to put a stop to this multiplicity is to focus on arguments of equilibrium selection. The question of equilibrium selection has been a main point of interest among game theorists, and the proposed solutions have mainly been based on different modifications of the assumptions underlying classical game-theoretic models. These assumptions State on the one hand that the payoff structure of the game (and the economic fundamentals) are commonly known, and on the other hand that all players are perfectly rational, and certain about the behavior of the others in equilibrium.

Following the work of Harsanyi(1973), Selten(1975), and Harsanyi & Selten(1988), Carlsson&VanDamme(1993) begin the development of the global games literature, which seems able to eradicate the multiplicity of equilibria. In this paper, the authors analyze the equilibrium selection in 2×2 games of incomplete information, where the payoff structure is determined by a random draw from a known class of games, and where the two players observe a noisy value of this draw. Observing independent noisy values of the true game enable to get rid of the perfect symmetry among players which is at the core of the problems of multiplicity. They show that, when this game induces two strict Nash equilibria, there is only one equilibrium remaining given the noisy observations of the players, and that this equilibrium is the risk-dominant one, which may not be Pareto-preferred to the other.

The global games literature has then been widely developed, under the impulse of MORRIS&SHIN. In their papers dealing with global game theory and different applications (MORRIS&SHIN(2003,2004A,2004B)), they extended the consequences of the paper of CARLSSON&VANDAMME(1993) to many different cases, and classes of games. They mainly studied games exhibiting strategic complementarities[1], where a continuum of players, has a choice to make between two actions.

The major applications that have been explored by now include currency crises, bank runs, or firm debts crises Nonetheless, until now, very few papers have tried to reconciliate the global games literature with models of sovereign debt crises.

Most papers based on global games, who tried to develop a model of sovereign debt crisis generated very simple, ad hoc models of sovereign debt roll-over. This is the case for instance in CARLSON&HALE(2005): the game exhibited in this paper may be basically reduced to a standard coordination game of regime change. If a sufficiently high fraction of the population accepts to buy bonds, the State will have the opportunity to live an additional period, and not to default, otherwise the State will be forced into default. The analogy can be translated into any other regime change issue without fundamental modifications of the model.

One way to make such a model more interesting, and more anchored in sovereign debt problematic, is the one that has been adopted by HATTORI(2004). In this paper, a first trade-off between a well developed model, and equilibrium selection using global games tools, has been built. The major improvement brought by this paper stands in the development of the market micro-structure: the market is segmented between specialists, who buy the bonds directly from the government, and the final investors, who buy the bonds from the specialists. Nonetheless, in this framework, the return on the government bonds is not at stake in the coordination game, which is at the opposite of the mainstream analysis of sovereign debt models.

In the classical, simple debt models of coordination, the value of the return on government bonds was exogenously set, while in the work of HATTORI(2004), it has been kept apart from the analysis. In this paper, we try to put the interest rate back at the core of the analysis of the sovereign debt models - as it was

[1]We say that a game exhibit strategic complementarities, when the incitation of a player to choose one action is monotonically increasing with the number of other agents who chooses this action

the case in the micro-founded, multiple-equilibria sovereign debt models - while attempting at the same time to introduce some sand in the wheels of the game, in order to select a unique equilibrium.

To do this, we start in the next section by developing a sovereign debt model that will serve as a framework for the following of the study. In section 3, we adapt the classical global games approach to this model, in order to get rid of the multiplicity of equilibria in a classical, though basic way. We then try in the following sections to get the model richer and more realistic. In section 4 we add a first assumption that creates a first link between the State of the economy, and the value of the return on the government bonds. In the last section, we set an asset market before the classical coordination game, in order to endogenize the value of this return into the game.

Our work is at the crossroads of the two literatures that we have presented above. The mainstream literature on sovereign debt crises, in particular the works of CALVO, and ALESINA&AL. have inspired us in building the framework of the analysis, developed in the first section, and inspired us in looking for an extension that could enable us to get rid of the multiplicity of equilibria in such models. Nonetheless, the literature we have the most relied on, in technical terms, has been the global games literature. We mainly rely on the previous works of CARLSON&VANDAMME, and MORRIS&SHIN in order to solve the different models that we set up in the first sections (section 3 and 4). For the last section, where we included an asset market to our game, the works of HELLWIG&AL(2004,2006) that dealt with currency crises have appeared to be very reliable starting points for an analogy in terms of sovereign debt crises.

2 The model set-up

We consider an open economy ruled by a government, and populated by a continuum of agents of mass 1. This open economy lives for two periods $t = \{0, 1\}$.

The government has to face some fixed expenses to run important works, pay fixed operating costs, or simply repay a past debt. In order to finance these expenses, the government chooses to borrow through the issuing of bonds. The remaining expenses would be financed by taxes raised at date 0. We assume that the value of the return on bonds is determined before the agents have the opportunity to buy them. We will see three different ways to determine the value of this return.

We will assume that, in case the fixed cost can't be fully financed after raising taxes and issuing debt, the government has to default, and doesn't reimburse its debt. Hereafter we will equivalently talk of *success of the project* for *no-default*, and *failure of the project* for *default*.

2.1 The government

1. In the first period, the government faces three types of expenses:

    - The government has fixed expenses, necessary to reimburse the past debt for instance, if the project succeeds (N_0). These expenses are exogenously set.
    - The government may pay subsidies, to the households (S_0), according to the State of the economy.
    - Finally, he has other possible expenses, that don't have a direct impact on the agents (G_0).

    On the other hand, it has three ways to be funded

    - He may sell bonds (B_0)
    - He also may raise taxes, according to the State of the economy (I_0).
    - In case of failure of the project, he may benefits from outside help (H_0) to help finance the expenses.

The government budget constraint in the first period is then:

$$N_0.1_{Success} + S_0 + G_0 = B_0 + I_0 + H_0 1_{Failure}$$

Or simply

$$N_0.1_{Success} + G_0 = B_0 + T_0 + H_0.1_{Failure}$$

Where T_0 summarizes all the transfers from the households, and can be either positive or negative.

Notice that the other possible expenses - G_0- behave as an adjustment variable. This value is observed by the government at the end of the first period, when all the other expenses have been made, and all the incomes have been perceived, in order to get a balanced first period budget constraint. The project fails whenever $N_0 \geqslant B_0 + T_0$.

2. In the second period, the government has two types of expenses:

   - The government may pay back the bonds he just issued, with some extra interests, to the households (RB_0), in case of no default in the first period.
   - It also may finance an outside help fund in case of failure (H_1).

   On the other hand, it has one way to be funded

   - It may raise taxes (T_1)

   The government budget constraint in the second period is then:

$$RB_0 1_{success} + H_1 1_{failure} = T_1$$

2.2 The agents

There is a continuum I of risk neutral agents of mass 1: $\int_I di = 1$.

We assume that their utility relies exclusively on their consumption in period 1, without any discounting rate:

$$\forall i \epsilon I,\ u_i(c_i^0, c_i^1) = c_i^1$$

Moreover, we assume that they consume all their money left in the second period.

The timing of their actions is as follows:

1. Each agent is initially endowed with N units of money at the beginning of the first period.
2. He then has the choice of buying N_0 units of public debt[2] (price 1, return factor R in case of no-default, 0 otherwise), or of buying exclusively riskless assets (price 1, return factor 1), knowing that the value of T_0 has already been determined.
3. He will then pay its first-period taxes and receive its subsidies, which sums to T_0.
4. In the second period, if he bought the bonds, he will receive his payment (RN_0), in case of no-default.
5. In case of default, he won't receive his payment if he chose to buy government bonds.
6. He will then pay taxes, T_1

The matrix of utility is the following[3]

	No default	Default
Buy N_0 units of debt	$N - N_0 - T_0 + RN_0 - T_1$	$N - N_0 - T_0 - T_1$
Buy N unit of riskless asset	$N - T_0 - T_1$	$N - T_0 - T_1$

Table 1: Table of utilities

[2]The agents can't buy less than N_0 units of bonds, unless they don't buy any. This setting is analogous to the paper of CALVO.

[3]We stood fuzzy about the determination of R, and T_1 in the general framework. Different models will be studied where these two determinations will be more detailed

2.3 Discussion

In different self-fulfilling debt crises models, the agents have a perfect foresight of the future, and of the reaction functions of the government (as in CALVO or ALESINA&AL.). In such models, the strategic player is generally the government. In our model nonetheless, we must be able to introduce some uncertainty in the beliefs of the agents - which is at the core of the global games literature -, and this uncertainty should concern only one variable, namely the fundamental of the economy. Moreover, the global games theory focuses on the choices of the agents, and on the coordination among them. Hence, this is important to make the agents the strategic players. The structural differences between self-fulfilling debt crises models and global games models have made us choose some specific assumptions.

In order to build this model, the assumptions we made is that the choices of the agents depend mainly on the period 0 value of taxes, not on the one levied in period one. Hence, the State faces a refinancing problem when T_0 is too low for $B_0 + T_0$ - which is bounded by $N_0 + T_0$ - to exceed N_0. If it may seem very different from the self-fulfilling debt crises models, it is more closely related to the assumptions made in the global games models of regime change, which makes the model easier to solve. The other point is that, trying to adapt a simple and intuitive model where the agents buy bonds if they expect the government to be able to reimburse them in the next period, appears to simply erase the existence of equilibrium.

But the reasons for such a choice are not purely technical. The usual assumption that we find in self-fulfilling debt crises models - the assumption that the agents make their decisions according to the value of the taxes in period 1, or according to the actions of the government in period 1 - would be unrealistic in practice: HATTORI states that the agents in practice, don't focus on what will occur one year ahead, but only 3 to 6 months ahead. Nonetheless, in order to make a stronger link between these two literatures, we chose in the last section to also take into account the value of the period 1 taxes in the decisions of the agents.

The framework that has been set up yields an interesting trade-off between a model that allows for the existence of equilibrium, and that is quite easy to solve, and a model that would be sufficiently realistic. The model here has some interesting implications, with respect to a classical regime change model.

The first change we made, was trying to give a reasonable meaning to the expression "fundamentals of the economy". This notion that we find in every paper dealing with global games, is quite easily understandable, even though it may seem a tough assumption to suppose that all the health of the economy can be summarized in a one-dimensional parameter. The choice we made here is to assume that T_0can catch all the necessary information for the creditors to make their decision: to us a high $T_0$is associated to a high value of the fundamentals, as it captures the fact that the population doesn't need subsidies and is able to pay high taxes, or the fact that the State is strong enough to have the ability to levy taxes. On the contrary, a low T_0 is associated to a poor value of the fundamentals.

Secondly, it gives some reasons for the State to need borrowing: here, we stated this reason as *reimbursing existing debt*. This assumption makes the model quite consistent with respect to the other existing models of debt roll-over, and makes this model realistic in terms of the analysis of sovereign debt crises: this is the impossibility of reimbursement of past debt that is at the origin of such crises. Moreover this makes this model embeddable into a many-period model. The hypothesis of full default in case of inability to reimburse the past debt is easily understandable: when the State defaults, not only the past issued bonds, but also the newly issued ones are not paid back. This corresponds to a practice that we find in reality, the cross-default, and this has been used in many similar sovereign debt models (see HATTORI(2004) for instance).

The third point that seemed interesting to us is that the success of the project enables the reimbursement of the bonds, according to a predetermined condition: $RB_0 = T_1$, where T_1 is variable, and may depend on R, N_0,.. according to the setting of the model. This simple feature enables - if we wish - to put some intertemporal consideration, as we will do in the last section, in order to determine a value of R consistent with future gains. Indeed, this value of R could take into account expected future incomes of the government. Until now, all the models that have incorporated global games in models of sovereign debt are lacking this aspect, which lowers the link between the interest rate and the fundamentals of the economy, and diminishes the role of the interest rate on the occurrence of default.

Having set up a general framework to study sovereign debt crises we will now try and solve this model with different assumptions.

3 A simple model

In the first part, we assume that the agents only know that they would be reimbursed if the project succeeds (i.e. if the State doesn't default). They are not interested to learn about the true value of T_1, as long as they know that T_1 is large enough to enable a full reimbursement in case of success of the project. One way to see this is to assume that T_1 is determined after the agents buy the bonds, in order to fulfill the condition $RB_0 = T_1$, and that the government is truthful, so as to be able to promise the reimbursement in case of no-default.

Moreover the value of R is set as exogenous. One way to see this is to say that it is determined by the agents, before buying the bonds, and before knowing anything about T_0. The value of R could then be determined by an ex-ante no-arbitrage condition.

For the sake of simplicity, we will assume that the agents have an improper prior about T_0: the ex-ante probability of observing any value of T_0 follows a uniform distribution over the entire real line.[4]

The timing of the game is summarized here:

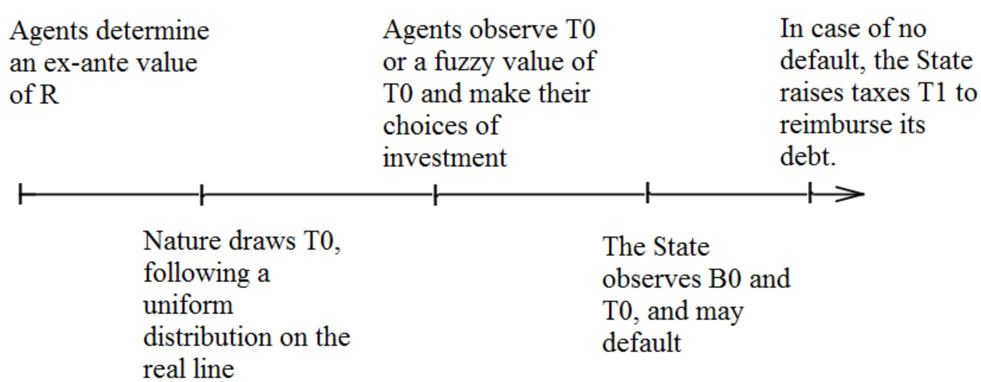

Figure 1: Time-line of the simple game

The game we are studying corresponds then to a coordination game between agents, as can be seen in the global game literature; the only question that remains is: will there be enough agents that are ready to buy bonds in order to avoid default?

3.1 The full information case

We will first assume that there is full information: the agents observe directly

[4]If this assumption may appear unrealistic, it has the advantage of generating less technical difficulties for the following. Moreover this assumption is not completely essential, and doesn't drive the results that we will observe (cf the discussion of section 4 to see why)

the true value of T_0, at the beginning of the game (i.e. before choosing whether or not buying bonds).

Before, we may assume that the value of R is determined through an ex-ante no arbitrage condition. This is a purely illustrative aspect, and doesn't have any realistic foundation.

The no-arbitrage condition yields: $Pr(N_0 \leqslant B_0 + T_0) * R = 1$, hence

$$\begin{aligned} R &= \frac{1}{Pr(N_0 \leqslant B_0 + T_0)} \\ &\in \left[\frac{1}{Pr(N_0 \leqslant T_0)}; \frac{1}{Pr(0 \leqslant T_0)}\right] \\ &= 2 \end{aligned}$$

Let's now turn to the game, and assume that the agents perfectly know the true value of T_0.

Three cases have to be distinguished:

1. $T_0 > N_0$: in that case, we always have $T_0 + B_0 > N_0$. This means that the project always succeeds. As $R > 1$: each player has a dominant strategy to buy bonds.

2. $T_0 < 0$: in that case, we always have $T_0 + B_0 < N_0$, and the project always fails. As $N - N_0 - T_0 - T_1 < N - T_0 - T_1$, each player has a dominant strategy not to buy bonds.

3. $T_0 \in [0, N_0]$: there are two symmetric pure strategy Nash equilibria. Either all the agents buy bonds, and $B_0 = N_0$, which leads to the success of the project, and maximizes the welfare of everyone. Or all the agents refuse buying bonds: $B_0 = 0$, $T_0 + B_0 < N_0$, the project fails, and the agents are in a second symmetric Nash equilibrium -though welfare-dominated by the first one. Notice that there exists also an infinity of other asymmetric pure Nash equilibria. [5]

To summarize, denoting D the proportion of agents who buy bonds, we plotted below with a green and a red line, the values of T_0 for which having the corresponding proportion of agents buying bonds lead to an equilibrium.

[5]As for now, when it will be time for interpretations, we will mainly focus on these values of T_0. Nonetheless, focusing only on these values while solving the game is not possible: as we will see, the existence of the two other dominance regions is necessary for getting rid of the multiplicity.

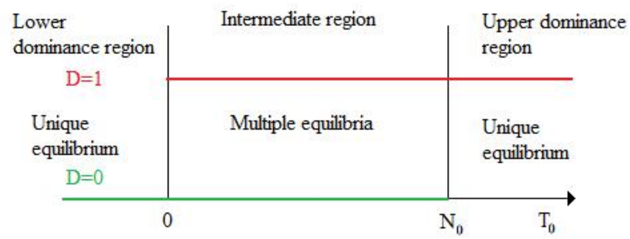

Figure 2: Dominance regions of the full information case

In this game, as in the other models of debt crisis such as Calvo, we have to cope with the problem of multiplicity of equilibria. For certain value of T_0, or certain states of the world, the agents have for instance the opportunity of coordinating on a welfare improving equilibrium, or on a welfare dominated equilibrium. The multiplicity of equilibria results from agents' inability to coordinate their choices in this strategic environment. This raises the interesting, and complicated question of the selection of the equilibrium: what would bring the agents to coordinate on such or such equilibrium? How could we anticipate the default of the State?

The path we are following is the global games theory, which tries to search an answer to this question in the incompleteness of information.

3.2 A simple game of incomplete information

We assume now that T_0 is not commonly known, and not perfectly observable by the different agents.

Instead, each agent $i \in I$ observes a "blurry" value of T_0, the value $T_0^i = T_0 + \varepsilon^i$, where the idiosyncratic noise ε^i is normally distributed, with mean 0, and precision β: $\forall i \in I$, $\varepsilon^i \sim \mathcal{N}(0, \beta^{-1})$. In our case, when the value of the tax, and subsidies have been decided, the different agents don't know precisely what will be this value before having to pay. As before, the prior about T_0 follows a uniform distribution about the entire real line.

We still consider that R is determined by the agents, before their observation of T_0^i, and that they know that T_1 will be high enough to enable the reimbursement of any stock of bonds in the next period.

We will try to see in that case, whether the multiplicity of equilibria still holds and solve for equilibrium.

3.2.1 Definitions

Let's first begin with some definitions

1. A *strategy* is a function specifying an action for each possible private signal. In this case, the set of actions is $\{Not\, buying\, bonds, \, Buying\, bonds\}$, which is designed hereafter as $\{0,1\}$: a strategy for agent i is then a function

$$\begin{cases} s : \mathbb{R} \rightarrow \{0,1\} \\ \quad T_0^i \rightarrow \{0,1\} \end{cases}$$

.

2. A *switching strategy* is a strategy where the agent buys bonds - and takes the more risky action if and only if he observes a private signal above some threshold. The switching strategy around $\hat{T}_0$ is thus the following:

$$s(T_0^i) = \begin{cases} 1 \; if \; T_0^i > \hat{T}_0 \\ 0 \; if \; T_0^i \leqslant \hat{T}_0 \end{cases}$$

.

3. An *equilibrium* is a profile of strategies - one for each agent - such that each agent's strategy maximizes his expected payoff conditional on the information available, when all other agents are following the strategies in the profile. Treating each possible realization of i's signal as a possible *type* of this agent, we are trying to solve for the Bayes Nash equilibria of the new game of imperfect information.

3.2.2 Switching strategy

In solving the model, looking for the equilibria, we will focus on switching strategy equilibria. If the restriction to such strategies is not without loss of generality, the switching strategy equilibria have an interesting property, as for the analysis of the uniqueness vs. multiplicity of equilibria: It turns out that whenever there is a unique switching strategy equilibrium, there is also a unique overall equilibrium. This property, whose proof is purely technical, will be demonstrated in appendix.

We first focus on switching strategy equilibria:

Agent i observes a private signal $T_0^i = T_0 + \varepsilon^i$. Each $\varepsilon^i$is independently normally distributed with mean 0, and variance β^{-1}(or precision β).

Hence as T_0 is assumed to be drawn from the real line with each realization equally likely, an agent who observes T_0^i considers T_0 to be distributed normally, with mean T_0^i, and variance β^{-1}.

Consider a player who has observed signal T_0^i and thinks that all his opponents are following the switching strategy with cutoff point $\hat{T}_0$. For $\hat{T}_0$ to be an equilibrium switching point, an agent whose posterior belief is $\hat{T}_0$ should be exactly indifferent between buying bonds and not buying bonds.

Let's denote p the expected probability of success of the project for this agent. His expected gain in utility to invest will be:

$$[p \times R - 1] \times N_0$$

[6]

Moreover, with T_0* the critical value of T_0 at which the State is on the margin between defaulting and not defaulting we have the following condition:

$$\begin{aligned} p &= Prob(T_0 \geqslant T_0 * |T_0^i) \\ &= Prob(T_0^i - \varepsilon^i \geqslant T_0 * |T_0^i) \\ &= Prob(\varepsilon^i \leqslant T_0^i - T_0 * |T_0^i) \\ &= \Phi(\sqrt{\beta}(T_0^i - T_0*)) \end{aligned}$$

Where $\Phi(.)$ stands -hereafter - for the c.d.f of $\mathcal{N}(0,1)$.

For an agent such that $T_0^i = \hat{T}_0$, we have: $\Phi(\sqrt{\beta}(\hat{T}_0 - T_0*)) = \frac{1}{R}$.

This first condition is summarized here:

[6] $p \times (N - N_0 - T_0 - T_1 + RN_0) + (1-p) \times (N - N_0 - T_0 - T_1) - [p \times (N - T_0 - T_1) + (1-p) \times (N - T_0 - T_1)]$

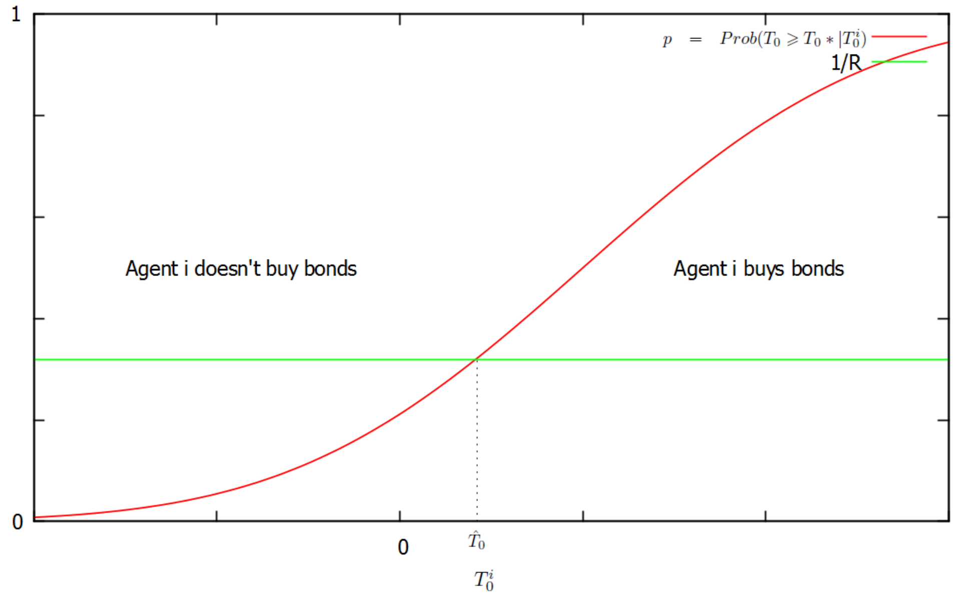

Figure 3: Determination of $\hat{T_0}$

In the equilibrium, we also have another condition:

The critical value of the fundamentals at which the State is on the margin between defaulting and not defaulting is at the state T_0* for which $T_0* = N_0(1-D)$, where D is the proportion of agents who buy bonds, resulting from the switching strategy around $\hat{T_0}$, when $T_0 = T_0*$.

We better understand how to find this value graphically.

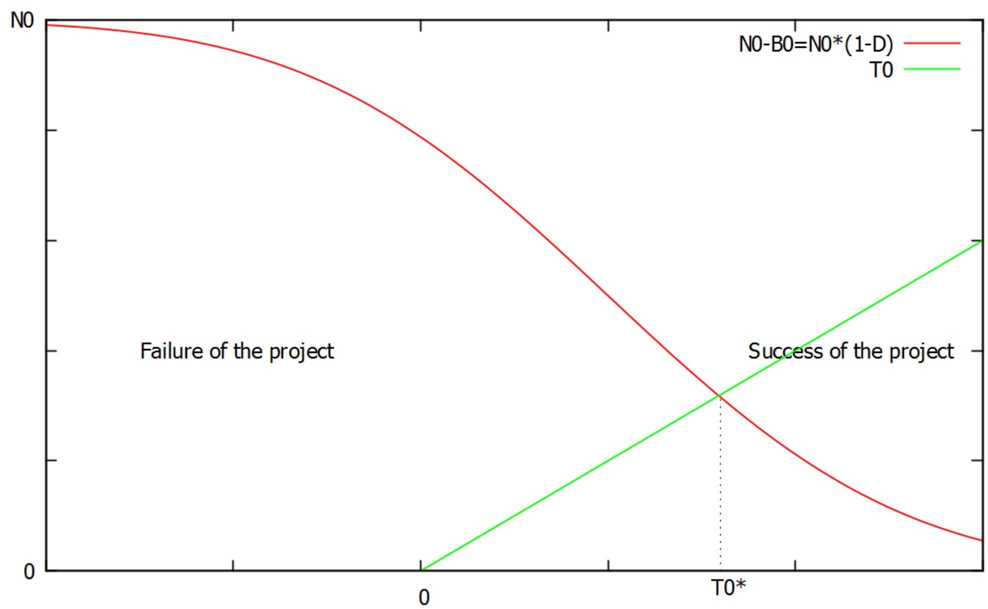

Figure 4: Determination of T_0*

We have

$$D = \int_I Prob(T_0^j \geqslant \hat{T}_0)dj = \int_I Prob(T_0 * - \hat{T}_0 \geqslant -\varepsilon^j)dj = \Phi(\sqrt{\beta}(T_0 * - \hat{T}_0))$$

Hence

$$1 - \Phi(\sqrt{\beta}(T_0 * - \hat{T}_0)) = \Phi(\sqrt{\beta}(\hat{T}_0 - T_0 *)) = \frac{T_0 *}{N_0}$$

Combining these two results enables to solve for the switching strategy equilibria

We have:

$$\begin{cases} T_0 * = \frac{N_0}{R} \\ \hat{T}_0 = \frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + \frac{N_0}{R} \end{cases}$$

For any $R > 1$, there is a unique associated switching strategy equilibrium.

For instance, assuming that the government bond is priced ex ante with respect to the riskless asset, we will have $R = \frac{1}{Pr(T_0 \geqslant T_0*)} = 2$, and the associated unique switching strategy equilibrium.

3.2.3 Uniqueness of equilibrium

So far, we have focused on the uniqueness of switching strategy equilibria. In fact, we can show that if there is a unique symmetric equilibrium in switching strategies, there can be no other equilibrium.

We will sketch the proof in Appendix for this model, but the proof is analogous for the two following extensions of the model. Hence, hereafter, we will simply focus on determining the uniqueness of switching strategy equilibria, as this is a sufficient and necessary condition for the uniqueness of equilibria.

3.3 Analysis of the model

This first model shows a classical analysis in terms of global games, and enables to understand how, the introduction of noise in the signals perceived by the agents enable to get rid of the multiplicity of equilibria. In a way, this first approach lets us understand how the selection works, and how the technical difficulties can be solved. In this first approach, to each value of R, corresponds for all the agents, a unique strategy where they decide to buy or not the government bonds according to the value of the perceived signal. This also gives a unique associated threshold value of T_0, $T_0 * (R)$ above which the project succeeds, and the State doesn't default, and under which the State is forced to default.

This is interesting to see that this uniqueness result holds in the limiting case as the noise vanishes $(\beta \to \infty)$. At first glance this may seem puzzling, as the private signal reveals in this case the true value of the fundamental. As analyzed by MORRIS&SHIN(2004B), if the *fundamental uncertainty (*uncertainty about the true value of the fundamentals) disappears, the multiplicity of equilibrium still doesn't hold due to the effect of the remaining *strategic uncertainty*, which is the uncertainty concerning the actions of the others. This strategic uncertainty then prevents the different agents to perfectly coordinate into any equilibrium.

The question that remains unexplained is: why does this strategic uncertainty hold? One idea to understand this is developed here.

Every agent, if some strategic uncertainty remains, will have a diffuse idea of the actions of the others, and be in a way more myopic concerning this. If we consider the proportion D of players who buy bonds, the subjective density function over [0;1] (i.e. the density function of D given T_0^i) will appear diffuse if there is lot of strategic uncertainty, and more concentrated otherwise. It can be shown that, this density is uniform for the switching player, even in the limiting case[7]. This suggests that *strategic uncertainty* remains in the limiting case, which enables the equilibrium selection.

Let us sketch this argument. Suppose that an agent is indifferent between buying government bonds or not, having observed signal T_0^i. Consider the probability that proportion p or less of the other agents have received a signal higher than this signal; p is the proportions of agents who will buy bonds. Let's denote $T_0 * (p)$ the marginal state such that, at $T_0 * (p)$, the proportion of agents with a signal higher than the critical level is exactly p.

We then have: $p = Pr(T_0^j \geq T_0^i) = Pr(T_0 * (p) + \varepsilon^i \geq T_0^i) = \Phi(\sqrt{\beta}(T_0 * (p) - T_0^i))$

What is now the probability for the true T_0 to be lower than $T_0 * (p)$ given T_0^i? This is: $Prob(T_0 \leq T_0 * |T_0^i) = Prob(T_0^i - \varepsilon^i \leq T_0 * |T_0^i) = \Phi(\sqrt{\beta}(T_0 * - T_0^i)) = p$.

Hence, as agent i is the switching agent,

$$\begin{array}{rcl} Pr(D \leq p|T_0^i) & = & Pr(\int_I Pr(T_0^j \geq T_0^i)dj \leq p|T_0^i) \\ & = & Pr(\Phi(\sqrt{\beta}(T_0 - T_0^i) \leq \Phi(\sqrt{\beta}(T_0 * (p) - T_0^i))|T_0^i) \\ & = & Prob(T_0 \leq T_0 * (p)|T_0^i) \\ & = & p \end{array}$$

The subjective cumulative density function of D for this agent is then the identity function, which shows that for this agent, the subjective density function of D is uniform. This tends to show in a way how, even when the fundamental uncertainty vanishes, the strategic uncertainty still holds.

Another interesting point of this first model, is that we can see that this uncertainty forces the State to default for values of $T_0 \in [0; \frac{N_0}{R}]$, where, in the perfect information case, the agents had the ability to coordinate in order to prevent this. On the other hand, it ensures that the State will survive for values of $T_0 \in [\frac{N_0}{R}; N_0]$. Let's focus on values of $T_0 \in [0; N_0]$. We can for instance assume that the State has the power to impose T_0 to stick into this range, even if the population keeps a uniform prior on the whole real line about T_0. In case of bad economical situations, where T_0 is low, still positive, the State must pay a high return in order to enable the project to succeed, and to avoid default: there will always be a value of R such that the project may take place, and the worse the economical situation, the higher the minimal value of R. Not surprisingly, a high R associated to bad economics outcomes and no default seems consistent.

[7] With a normal prior on T_0, the subjective density function would have been uniform only in the limiting case

Nonetheless, in this first basic model, we haven't explicited any clear link between the value of R, and the value of the fundamentals: we simply stated that the fundamentals are determined by the Nature, and that the value of R is determined before the coordination game. The value of R may perfectly be uncorrelated to the value of T_0, and this doesn't seem quite realistic. We will then try to build a more direct link between these two variables in the following sections, in order to get a richer and more consistent model.

4 Toward a micro-founded model: R as an exogenous vector of information

In the first model, the value of R could be set up at any value, such that $R > 1$, and we have for instance priced it ex ante, before knowing anything about the state of the economy. This is not completely satisfactory: when you buy bonds, the return on the bonds gives you an indication on the state of the economy.

We will now consider in this first extension that when you observe the value of R, you have an insight on the health of the economy - which can be seen here as the ability of the government to raise taxes. Nonetheless, the actors of the game go on taking R as exogenous. As in the model of HATTORI(2004) we could consider that we focus on the coordination game of the *final investors,* behaving as price takers: they observe R, previously determined by *specialists,* and this value is trusted as partly revealing the true state T_0. They then make their decisions of investment, assuming that if the government doesn't default at the end of the first period they would be reimbursed with certainty.

The timing of the new game is represented below.

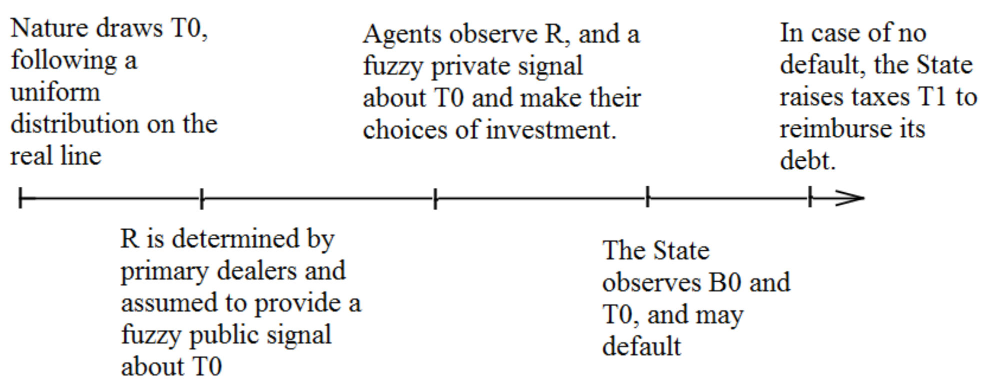

Figure 5: Time-line of the second game

4.1 The model set-up

We now assume that R behaves as a *public signal*: the value of R is publicly observed, and commonly known as reflecting the value of T_0 in this way $\gamma\Phi^{-1}(\frac{1}{R}) = T_0 + v$, where $v \sim \mathcal{N}(0, \alpha^{-1})$, and $\gamma \in \mathbb{R}_*^+$.

The γ term in the left-hand side enables to quantify the informational role of R: the public signal that the different agents observe is given by $z = \gamma\Phi^{-1}(\frac{1}{R}) =$

T_0+v, and we have $\frac{\partial z}{\partial \Phi^{-1}(\frac{1}{R})} = \gamma$. Hence, the higher the coefficient γ, the more a variation of $\Phi^{-1}(\frac{1}{R})$ impacts the public signal, and the higher the informational impact of R.

For the sake of simplicity we still assume that T_0 is drawn from an improper prior, and, for the same reasons as before, we focus on switching strategy equilibria. Once T_0 has been drawn, the agents observe the value of the public signal R, and their private signal T_0^i . Then, they decide whether or not to buy bonds. The way to solve for the switching strategy equilibria is analogous to the previous model.

4.2 Solving the model

Agent i observes a private signal $T_0^i = T_0 + \varepsilon^i$ - with each ε^i independently normally distributed with mean 0, and variance β^{-1}, and a public signal $\gamma\Phi^{-1}(\frac{1}{R}) = T_0 + v$, where $v \sim \mathcal{N}(0, \alpha^{-1})$.

The introduction of the public signal gives additional information on the true value of T_0.

Let's now take a look at the posterior distribution of T_0 (the updated belief) for the agent i, and denote f_X the probability density function of the random variable X.

Bayes theorem states that

$$f_{T_0}(t_0|\gamma\Phi^{-1}(\frac{1}{R}) = z, T_0^i = t_0^i) \propto f_{T_0^i}(t_0^i|\gamma\Phi^{-1}(\frac{1}{R}) = z, T_0 = t_0) \times f_{\gamma\Phi^{-1}(\frac{1}{R})}(z|T_0 = t_0) \times f_{T_0}(t_0)$$

We have

$$\begin{cases} f_{T_0}(t_0) = 1 \qquad (improper \quad uniform \quad prior) \\ f_{\gamma\Phi^{-1}(\frac{1}{R})}(z|T_0 = t_0) \propto \exp(-\frac{1}{2}\alpha(z-t_0)^2) \\ f_{T_0^i}(t_0^i|\gamma\Phi^{-1}(\frac{1}{R}) = z, T_0 = t_0) = f_{T_0^i}(t_0^i|T_0 = t_0) \propto \exp(-\frac{1}{2}\beta(t_0^i - t_0)^2) \end{cases}$$

Hence

$$f_{T_0}(t_0|\gamma\Phi^{-1}(\frac{1}{R}) = z, T_0^i = t_0^i) \propto \exp(-\frac{1}{2}\beta(t_0^i - t_0)^2 - \frac{1}{2}\alpha(z-t_0)^2)$$

And

$$\beta(t_0^i - t_0)^2 + \alpha(z - t_0)^2 = (\alpha+\beta)(t_0 - \frac{\alpha z + \beta t_0^i}{\alpha+\beta})^2 + \frac{\alpha\beta}{(\alpha+\beta)}(t_0^i - z)^2$$

.

The posterior distribution of T_0, given $\gamma\Phi^{-1}(\frac{1}{R})$ and T_0^i has then the form of a normal law, with mean $\frac{\alpha\gamma\Phi^{-1}(\frac{1}{R}) + \beta T_0^i}{\alpha+\beta}$, and precision $\alpha+\beta$.

Observing the public and the private signal, the agents have an updated signal about T_0, which is obtained as seen in the following figure.

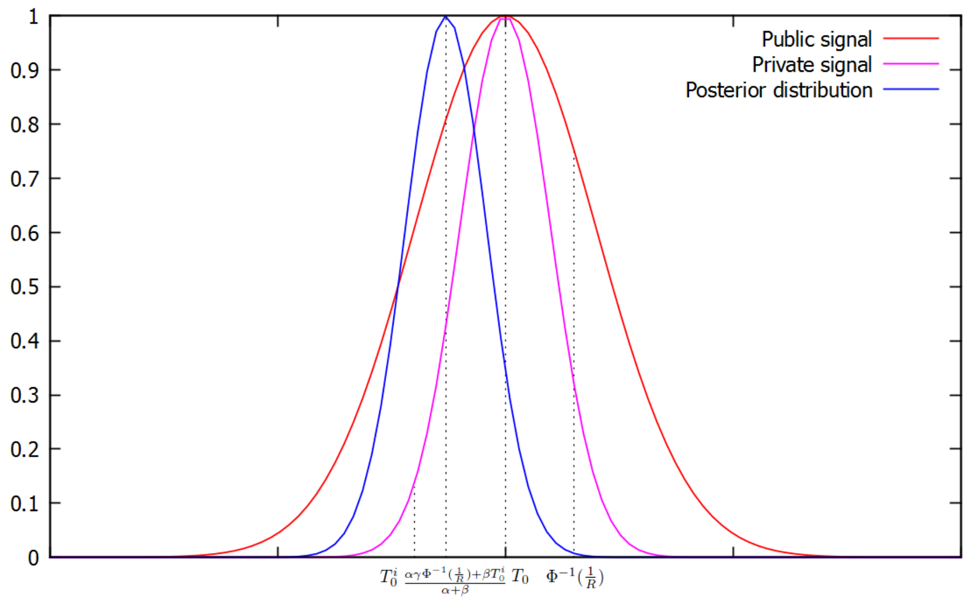

Figure 6: Determination of the posterior distribution

The demonstration that follows is then similar as the one above, with a new posterior distribution of T_0.

Let's again denote $\hat{T}_0$ the switching strategy cutoff point. For $\hat{T}_0$ to be an equilibrium switching point, an agent whose posterior belief is $\hat{T}_0$should be exactly indifferent between buying bonds and not buying bonds.

Let's denote p the expected probability of success of the project for this agent. His expected gain in utility to invest will be again:

$$p \times R - 1$$

Moreover, with T_0* the critical value of T_0 at which the project is on the margin between failing and succeeding we have the following condition:

$$\begin{array}{rcl} p & = & Prob(T_0 \geqslant T_0 * |T_0^i, \gamma\Phi^{-1}(\frac{1}{R})) \\ & = & Prob(\frac{\alpha\gamma\Phi^{-1}(\frac{1}{R})+\beta T_0^i}{\alpha+\beta} + \frac{1}{\sqrt{\alpha+\beta}}\pi^i \geqslant T_0 * |T_0^i, \gamma\Phi^{-1}(\frac{1}{R})) \\ & = & Prob(-\pi^i \leqslant \sqrt{\alpha+\beta}(\frac{\alpha\gamma\Phi^{-1}(\frac{1}{R})+\beta T_0^i}{\alpha+\beta} - T_0*)|T_0^i, \gamma\Phi^{-1}(\frac{1}{R})) \\ & = & \Phi(\sqrt{\alpha+\beta}(\frac{\alpha\gamma\Phi^{-1}(\frac{1}{R})+\beta T_0^i}{\alpha+\beta} - T_0*)) \end{array}$$

For an agent such that $\frac{\alpha\gamma\Phi^{-1}(\frac{1}{R})+\beta T_0^i}{\alpha+\beta} = \hat{T_0}$, we have:

$$\Phi(\sqrt{\alpha+\beta}(\hat{T}_0 - T_0*)) = \frac{1}{R}$$

.

At the equilibrium, we also have another condition:

The critical value of the fundamentals at which the project is on the margin between failing and succeeding is at the state T_0* for which $T_0* = N_0(1-D)$, where D is the proportion of agents who buy bonds, resulting from the switching strategy around $\hat{T_0}$, when $T_0 = T_0*$.

We have

$$\begin{array}{rcl} D & = & \int_I Prob(\frac{\alpha\gamma\Phi^{-1}(\frac{1}{R})+\beta T_0^i}{\alpha+\beta} \geqslant \hat{T_0})di \\ & = & \int_I Prob(T_0^i \geqslant \frac{(\alpha+\beta)\hat{T_0}-\alpha\gamma\Phi^{-1}(\frac{1}{R})}{\beta})di \\ & = & \Phi(\sqrt{\beta}(T_0 * - \frac{(\alpha+\beta)\hat{T_0}-\alpha\gamma\Phi^{-1}(\frac{1}{R})}{\beta})) \end{array}$$

Hence

$$1-\Phi(\sqrt{\beta}(T_0*-\frac{(\alpha+\beta)\hat{T_0}-\alpha\gamma\Phi^{-1}(\frac{1}{R})}{\beta})) = \Phi(\sqrt{\beta}(\frac{(\alpha+\beta)\hat{T_0}-\alpha\gamma\Phi^{-1}(\frac{1}{R})}{\beta} - T_0*)) = \frac{T_0*}{N_0}$$

Combining these two results enables to solve for the switching strategy equilibria.

We have:

$$\begin{cases} \Phi(\sqrt{\beta}(\frac{(\alpha+\beta)\hat{T_0}-\alpha\gamma\Phi^{-1}(\frac{1}{R})}{\beta} - T_0*)) = \frac{T_0*}{N_0} \\ \Phi(\sqrt{\alpha+\beta}(\hat{T}_0 - T_0*)) = \frac{1}{R} \end{cases}$$

Hence:

$$\begin{cases} \hat{T_0} = \frac{1}{\sqrt{\alpha+\beta}}\Phi^{-1}(\frac{1}{R}) + T_0* \\ \Phi(\frac{1}{\sqrt{\beta}}(\alpha T_0 * + \sqrt{\alpha+\beta}\Phi^{-1}(\frac{1}{R}) - \alpha\gamma\Phi^{-1}(\frac{1}{R}))) = \frac{T_0*}{N_0} \end{cases}$$

$$\Longleftrightarrow \begin{cases} \hat{T_0} = \frac{1}{\sqrt{\alpha+\beta}}\Phi^{-1}(\frac{1}{R}) + T_0* \\ \sqrt{\beta}\Phi^{-1}(\frac{T_0*}{N_0}) - \alpha T_0* = \sqrt{\alpha+\beta}\Phi^{-1}(\frac{1}{R}) - \alpha\gamma\Phi^{-1}(\frac{1}{R}) \end{cases}$$

4.3 uniqueness of equilibrium

The uniqueness of the equilibrium exclusively relies on the second equation. Having observed the signals, is there a unique threshold value for T_0 above which the government doesn't default? Is there a unique threshold value for T_0^i above which the agents buy bonds?

Let's make a change of variables.

Given z, $\begin{array}{cccc} \psi & : & x & \mapsto & \Phi^{-1}(\frac{x}{N_0}) \\ & & [0; N_0] & \to & \mathbb{R} \end{array}$ is a bijection.

Solving for the uniqueness of equilibrium of this problem, corresponds to solve for the uniqueness of $x \in \mathbb{R}$ such that:

$\sqrt{\beta}x - \alpha N_0 \Phi(x) = \sqrt{\alpha+\beta}\Phi^{-1}(\frac{1}{R}) - \alpha\gamma\Phi^{-1}(\frac{1}{R})$

This sums up to study the function g:

$\begin{array}{cccc} g & : & x & \mapsto & x - \frac{\alpha N_0}{\sqrt{\beta}}\Phi(x) + c \\ & & \mathbb{R} & \to & \mathbb{R} \end{array}$ where c stands for $\frac{\sqrt{\alpha+\beta}}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) - \frac{\alpha\gamma}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R})$.

A simple limit analysis shows by arguments of continuity that the function admits at least one zero.

Let's now analyze the derivative of this function:

$\forall x \in \mathbb{R}, g'(x) = 1 - \frac{\alpha N_0}{\sqrt{\beta}}\phi(x)$

Two cases can be distinguished:

1. Either $\frac{\sqrt{\beta}}{\alpha N_0} > \phi(0) = \frac{1}{\sqrt{2\pi}}$, in which case we can ensure the uniqueness of the equilibrium.

2. Or $\frac{\sqrt{\beta}}{\alpha N_0} < \phi(0) = \frac{1}{\sqrt{2\pi}}$, in which case we can't ensure the uniqueness of the equilibrium: according to the value of c, hence the value of R, there can be multiple equilibria.

The three possible different cases have been represented in the figure below, where $g(a, b, .)$ is the function defined by:

$$g(a, b, .) : x \mapsto x - a\Phi(x) + b$$

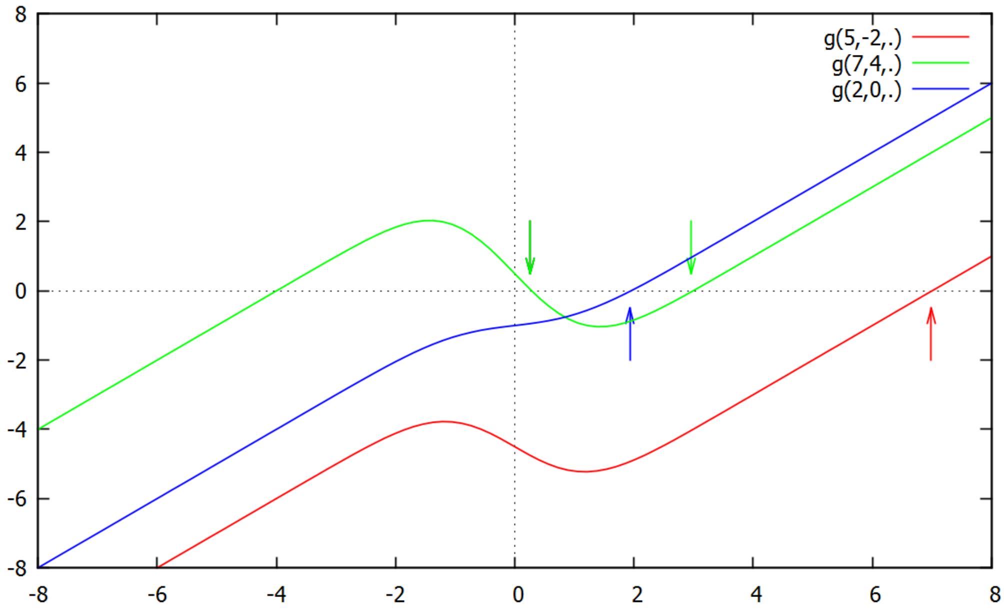

Figure 7: Multiplicity vs. uniqueness of equilibrium: Determination

For the sake of simplicity, we will keep in mind the sufficient conditions: there is a unique equilibrium if $\frac{\sqrt{\beta}}{\alpha N_0} > \frac{1}{\sqrt{2\pi}}$ and there may be multiple equilibria otherwise.

This condition is represented in the following figure, for $N_0 = 1$

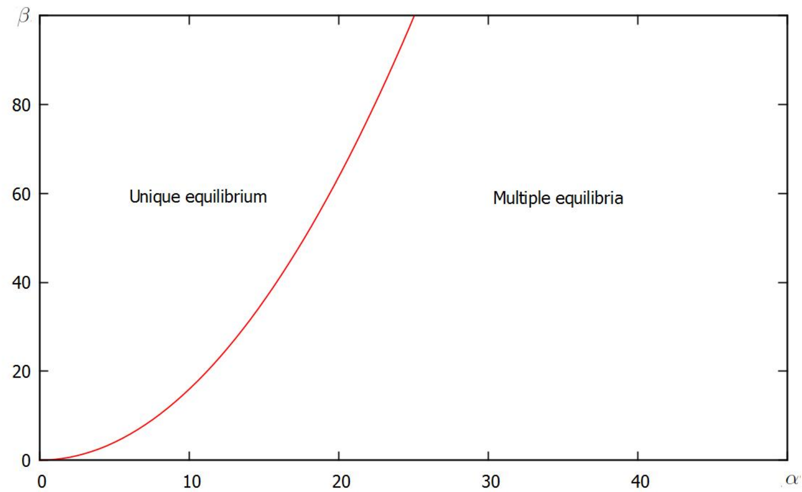

Figure 8: uniqueness vs. Multiplicity of equilibria: Conditions on the precisions

4.4 Analysis of the model

In this second framework we still kept the determination of the interest rate away from the game. We assumed that this determination occurs after T_0 has been drawn, but before the agents choose whether or not buying bonds. We also assumed that the determination of this value could help the investors in this decision by giving them an additional public signal about the underlying fundamental. How can we interpret the introduction of such a signal?

One way to see this is to assume that the return on this asset is determined according to the value of the fundamentals, through an auction amongst big investors in a primary market. This market segmentation is consistent with the tender of the debt in many countries. For instance, the tender of the French debt is determined in an analogous way: there are 20 specialists - French and foreign banks and institutions - who have the mission, and the right to buy the bonds sold by the government, at a return determined by auction. Their role is mainly to make these auctions possible (each of them must buy at least

2% of each kind of the bonds issued), and then to ensure the fluidity of the secondary market to whom they sold the debt that they bought. They also have a counseling role.

This return reflects in a way the state of the economy: the weaker the economy, the higher the return, as the primary dealers fear the bankrupt of the State. The way of auctioning the French debt - *adjudication à la hollandaise* - is as follows: the different primary dealers propose some price (or return), and quantity of bonds, and the *Agence France Trésor* then sells the bonds at the proposed price, beginning with the higher offer, and ending when they have issued enough bonds.

Nonetheless our model does not completely reproduce this market segmentation. A better interpretation would be to imagine that the primary dealers only set the price of the asset [8], but that the agents decide whether or not buying bonds directly to the State: hence, if an insufficient number of agents buy bonds, the State itself declares default, not the primary dealers.

In this new model, the essential improvement is the addition of *public information* provided by the returns. The introduction of the public information in the model has some interesting implications.

The first ones are more of technically order: it enables to solve the model when an additional public signal is at stake, and to understand how the introduction of such a signal modifies the beliefs of the agents on the state of the economy. It will have important implications in the next section. It also enables in a way to understand how a normal prior on T_0 could have changed the previous results (compared to an improper, uniform prior). We understand that it would basically add some technical difficulties, and a restriction on the precision of this prior to get the previous results; more precisely, the uniform prior assumption appears to be a limiting case of a normal prior assumption when the precision of the normal distribution tends to 0.

It also enables to understand the implications of an informative meaning of R. As in the previous model, this new public signal generates both *strategic* and *fundamental uncertainty.* This, again can be seen in the limiting case of the precision of the signal. Let's consider the limiting case: $\alpha \to \infty$,$\beta \to \infty$ and

8

This is actually one of their role: they must give a price of the asset, and a corresponding quantity, they are ready to buy or sell on the secondary market, in order to fluidize it.

$\frac{\alpha N_0}{\sqrt{\beta}} = c < \sqrt{2\pi}$. We then get:

$$\begin{cases} \hat{T_0} = T_0 * \\ \Phi^{-1}(\frac{T_0*}{N_0}) - c\frac{T_0*}{N_0} = \Phi^{-1}(\frac{1}{R}) - \frac{c\gamma}{N_0}\Phi^{-1}(\frac{1}{R}) \neq \Phi^{-1}(\frac{1}{R}) - \frac{c}{R} \end{cases}$$

We notice that the value of R appears in this expression in a very different way than in the previous model. This is quite puzzling, for two reasons: the first one is that knowing that the value of R reflects perfectly the fundamental T_0 shouldn't yield any additional information to the players, who already know that their private signal perfectly reflects the true value of the fundamental; the second one is that, the information contained in the public signal is still much less precise as the public signal, and then the information given by the public signal should be dominated by the private signals. This shows the importance of the strategic uncertainty in the outcome of such games: the value of R has an important role in making inferences concerning the (updated) beliefs of other creditors, and then, on what they do.

In this model, R has two opposite roles: on the one hand, an increase in R - or a decrease in $\frac{1}{R}$- generates a decrease in the public signal $\Phi^{-1}(\frac{1}{R})$, which tells the agents that the fundamentals of the economy are worse, and disincentives them from buying bonds. We call it the *informative effect* of the return on the bonds. In the previous couple of equations, the informative effect is symbolized by the term $\frac{c\gamma}{N_0}\Phi^{-1}(\frac{1}{R})$. On the other hand, an increase in R leads to an increase in the expected payoff for any agent, p being fixed. We call it the *payoff effect* of the return on the bonds. In the previous couple of equations, the payoff effect is symbolized by the term $\Phi^{-1}(\frac{1}{R})$. If the informative effect is higher than the payoff effect, a sufficiently high R will deter the agents from investing and lead to a default while otherwise, a low R would lead to a default. Consistent with the intuition, the payoff effect is stronger than the informative effect for high values of N_0 (as the expected payoff is much higher for a slight evolution of R), and the informative effect is stronger than the payoff effect for high values of γ, which is consistent with the definition of γ.

Under the light of this statement, we could analyze the last sovereign debt crisis in Europe as an information crisis: if a country has a very low R, and a low risk of default, this is because the informative effect is higher than the payoff effect. Hence, the determined interest rate is assumed by the final investors, to follow quite sharply the variations of the fundamentals: the slightest variation in T_0 would be immediately - and sensitively - transmitted into the value of R. When they suddenly notice that it is not the case, the agents update their informative effect, which falls below the payoff effect: hence, to limit the risk of default, the returns have to rise.

5 Toward a micro-founded model: R as an endogenous information vector

The previous model yields interesting implications both technically, with the introduction of a public signal, and in terms of interpretations of sovereign debt crises, by giving a first model where R is at the core of the default risk mechanisms. Nonetheless, there remain some points that call for an extension.

The first one is that the determination of R still remains unexplained in these previous models, and we would wish to integrate this price formation into a wider model.

The second one is the fact that this model doesn't have any forward-looking component; R is assumed to reflect, more or less starkly the state of the world at time 0, while the return on the bonds will be paid at time 1. This would make our model more realistic to take into account this intertemporality, as the different multiple-equilibria, micro-founded debt models do (Calvo, Alesina,..).[9]

The previous model only paid attention to a stylized secondary market. In this model we will try to take into account the primary market too, and to pay attention to the agents as price-setters, and not as price-taker anymore.

5.1 The new game

We now add a prior asset pricing game to our model, in order to make it more complete.

At the beginning, the different agents interact in a Walrasian auction in order to determine the appropriate return on the bonds. This is symbolized by a supply-demand equilibrium, where the different agents coordinate on a value of R, having observed a blurry value of T_1. The supply-demand equilibrium then basically states that:

$$RDN_0 = T_1$$

where the functional form of T_1 will be explicited, and discussed below.

Once they have interacted in this asset market, the return that has been determined provides a public signal about T_0, and we get back to the question of solving a coordination game amongst agents, similar to the one we solved in the previous section.

The new timing of the game is the following.

[9] Hattori(2004) states that such an assumption of intertemporality is opposed to what we observe in practice. He says that when the investors worry about the solvency of the government, they would take into account if the government has enough finance for the coming quarter or half-year, implying that the concerns of the investors keep restricted to the near future. In this case, the previous model may be sufficient. Nonetheless, we trust that even if the investors are able to worry only up to the next half-year, the question of intertemporality should be left apart.

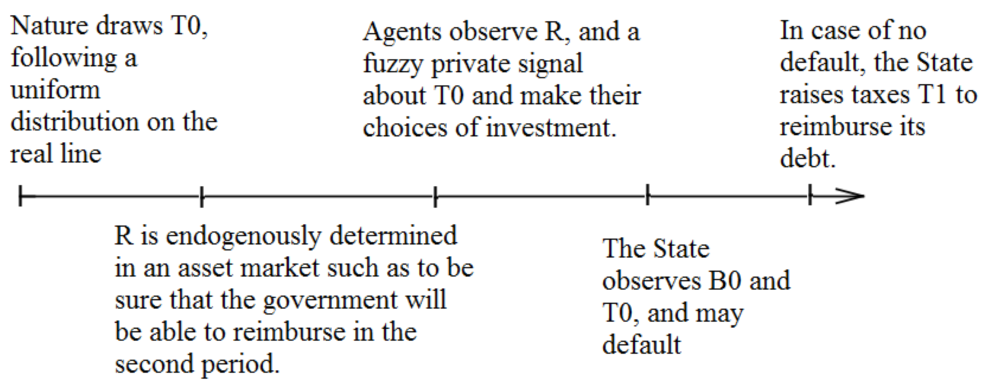

Figure 9: Time-line of the last game

5.2 Solving the model

The new game is a two-stage game. We first solve the asset pricing game, and then the coordination game.

5.2.1 The asset-pricing game

We assume now that the value of T_1 is determined at the beginning of the game, and is perceived by the different agents as:

$$T_1 = RN_0\Phi(s_0 + \Phi^{-1}(\frac{1}{R}) + \Phi^{-1}(\frac{1}{N_0}))$$

with $s_0 \sim \mathcal{N}(0, \delta^{-1})$. It enables to take into account the fact that the taxes have not been levied yet, and that there could be some variability between the expected taxes and the realized one. It also enables to avoid the asset market being fully revealing of the value of the fundamental, which would lead to multiplicity of equilibria. The form of the function simplifies the calculations and enables to take into account the effect of R and N_0 on the market clearing condition: for a given value of s_0, the amount of taxes that will be levied is defined. Hence an increase in R should be reflected by a downward pressure on the demand side: the total amount of money that can be paid back is fixed, increasing R reduces the number of bonds that can be reimbursed without jeopardizing the second period financial equilibrium. An increase in N_0 should have a symmetric impact given the market-clearing condition, and the economical reason for this is perfectly similar to the one we just stated.

Giving the functional form assumption, the market clearing condition states that:

$$RDN_0 = R\Phi(\sqrt{\beta}(T_0 - \hat{T}_0(R)))N_0 = T_1 = R\Phi(s_0 + \Phi^{-1}(\frac{1}{R}) + \Phi^{-1}(\frac{1}{N_0}))N_0$$

Or simply

$$\Phi(\sqrt{\beta}(T_0 - \hat{T}_0(R))) = \Phi(s_0 + \Phi^{-1}(\frac{1}{R}) + \Phi^{-1}(\frac{1}{N_0}))$$

Using the asset market clearing condition, the information conveyed by the asset price is summarized by

$$z = \hat{T}_0(R) + \Phi^{-1}(\frac{1}{N_0})\frac{1}{\sqrt{\beta}} + \Phi^{-1}(\frac{1}{R})\frac{1}{\sqrt{\beta}} = T_0 - s_0\frac{1}{\sqrt{\beta}}$$

where $\hat{T}_0(R)$ stands hereafter for the switching threshold of the *private signal*. $R(T_0, s_0)$ is an admissible solution in equilibrium, if and only if, for all T_0, s_0 the above condition is satisfied. The left hand-side of the above expression only depends on R, on which the agents can condition their bids, as in the previous models. The right hand-side, only depends on the unobservable of the game: T_0 and s_0. Hence, if the Walrasian auctioneer[10] conditions R on $z = T_0 - s_0\frac{1}{\sqrt{\beta}}$, choosing the same value of $R(z)$ for all T_0, s_0 such that $z = T_0 - s_0\frac{1}{\sqrt{\beta}}$, z becomes a sufficient statistic for the information conveyed by R about T_0 on the equilibrium path.

We assume that this is the case, and that the value of R is determined after observing a draw of z (the agents who take part to the game don't have the ability to disentangle T_0 from s_0 in their observation). The information conveyed by R about T_0 is then summarized by a public signal, z which is - conditional on T_0 - normally distributed with mean T_0, and precision $\beta\delta$. It is interesting to notice that the precision of the public signal here increases with the precision of the private signals: the more precise the exogenous private signals, the more precise the endogenous public signal becomes.

5.2.2 The coordination game

Once R has been determined, we can focus on the coordination game.

As in the previous model, the agents now observe both a private signal T_0^i, and a public signal $z = \hat{T}_0(R) + \Phi^{-1}(\frac{1}{N_0})\frac{1}{\sqrt{\beta}} + \Phi^{-1}(\frac{1}{R})\frac{1}{\sqrt{\beta}}$.

We keep the same two conditions as before. The first one states that an agent, whose posterior belief $\frac{\beta\delta z + \beta T_0^i}{\beta\delta + \beta}$ is exactly $\hat{T}_0(R) + \frac{\delta(\Phi^{-1}(\frac{1}{N_0})\frac{1}{\sqrt{\beta}} + \Phi^{-1}(\frac{1}{R})\frac{1}{\sqrt{\beta}})}{\delta+1}$,

[10] The Walrasian auctioneer is the hypothetical auctioneer that matches demand and supply in a perfect competition market

should be indifferent between buying bonds and not buying bonds. The same arguments as before yield:

$$\Phi(\sqrt{\beta+\beta\delta}(\hat{T_0}(R)+\frac{\delta(\Phi^{-1}(\frac{1}{N_0})\frac{1}{\sqrt{\beta}}+\Phi^{-1}(\frac{1}{R})\frac{1}{\sqrt{\beta}})}{\delta+1}-T_0*))=\frac{1}{R}$$

The second one states that, $T_0*(R)$ is the critical value of the fundamental at which the State is on the margin between defaulting and not. It is thus the state such that:

$$\Phi(\sqrt{\beta}(\hat{T_0}(R)-T_0*))=\frac{T_0*}{N_0}$$

.

We then have:

$$\begin{cases}\Phi(\sqrt{\beta+\beta\delta}(\hat{T_0}(R)+\frac{\delta(\Phi^{-1}(\frac{1}{N_0})\frac{1}{\sqrt{\beta}}+\Phi^{-1}(\frac{1}{R})\frac{1}{\sqrt{\beta}})}{\delta+1}-T_0*))=\frac{1}{R}\\ \Phi(\sqrt{\beta}(\hat{T_0}(R)-T_0*))=\frac{T_0*}{N_0}\end{cases}$$

Rearranging, we get:

$$\begin{cases}\Phi(\sqrt{1+\delta}(\frac{\delta(\Phi^{-1}(\frac{1}{N_0})+\Phi^{-1}(\frac{1}{R}))}{\delta+1}+\Phi^{-1}(\frac{T_0*}{N_0})))=\frac{1}{R}\\ \hat{T_0}(R)=T_0*(R)+\frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{T_0*}{N_0})\end{cases}$$

5.2.3 The full game

The conditions above lead to the following equilibrium characterization:

1.In equilibrium $T_0*(R)$ and $\hat{T_0}(R)$ are given by:

$$\begin{cases}\Phi^{-1}(\frac{T_0*(R)}{N_0})=(\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})\Phi^{-1}(\frac{1}{R})-\frac{\delta}{1+\delta}\Phi^{-1}(\frac{1}{N_0})\\ \hat{T_0}(R)=\Phi((\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})\Phi^{-1}(\frac{1}{R})-\frac{\delta}{1+\delta}\Phi^{-1}(\frac{1}{N_0}))+\frac{1}{\sqrt{\beta}}\left((\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})\Phi^{-1}(\frac{1}{R})-\frac{\delta}{1+\delta}\Phi^{-1}(\frac{1}{N_0})\right)\end{cases}$$

2. Any equilibrium asset price function is implicitly characterized by

$$z=\Phi((\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})\Phi^{-1}(\frac{1}{R})-\frac{\delta}{1+\delta}\Phi^{-1}(\frac{1}{N_0}))+\frac{1}{\sqrt{\beta}}\left((\frac{1}{\sqrt{1+\delta}}+\frac{1}{1+\delta})\Phi^{-1}(\frac{1}{R})+\frac{1}{1+\delta}\Phi^{-1}(\frac{1}{N_0})\right)$$

What we see here is that T_0* and $\hat{T_0}$ are uniquely determined for a given R. Nonetheless the new fact is that R is endogenously determined: this determination may lead to multiplicity of equilibria; the multiplicity may not arise from the second-stage coordination game, but from rational expectations equilibrium in the first-stage asset market.

The interest rate $R(z)$ is picked out of a correspondence $\hat{R(z)}$. The question of the uniqueness of the equilibrium then relies on the following question: is $\hat{R(z)}$ single-valued for all z or is there some values of z for which the correspondence admits multiple values?

5.3 uniqueness of equilibrium

The condition for the uniqueness is analogous to the one we already determined. The analysis remains the same. Let's make a change of variables.

Given z, $\begin{array}{cccc} \psi & : & R & \mapsto & x = (\frac{1}{\sqrt{1+\delta}} - \frac{\delta}{1+\delta})\Phi^{-1}(\frac{1}{R}) - \frac{\delta}{1+\delta}\Phi^{-1}(\frac{1}{N_0}) \\ & & [1;+\infty] & \to & \mathbb{R} \end{array}$ is a bijection if $(\frac{1}{\sqrt{1+\delta}} - \frac{\delta}{1+\delta}) \neq 0$.

Solving for the uniqueness of equilibrium of this problem, corresponds to solve for the uniqueness of x such that:

This sums up to study the function g:

$$\begin{array}{ccccc} g & : & x & \mapsto & \Phi(x) + \frac{(\frac{1}{\sqrt{1+\delta}}+\frac{1}{1+\delta})}{(\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})}\frac{1}{\sqrt{\beta}} \times x + (\frac{1}{\sqrt{\beta}}\frac{1}{1+\delta} + \frac{\delta}{1+\delta}\frac{(\frac{1}{\sqrt{1+\delta}}+\frac{1}{1+\delta})}{(\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})}\frac{1}{\sqrt{\beta}})\Phi^{-1}(\frac{1}{N_0}) \\ & & \mathbb{R} & \to & \mathbb{R} \end{array}$$

A simple limit analysis shows by arguments of continuity that the function admits at least one zero.

Let's now analyze the derivative of this function:

$\forall x \in \mathbb{R}, g'(x) = \phi(x) + \frac{(\frac{1}{\sqrt{1+\delta}}+\frac{1}{1+\delta})}{(\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})}\frac{1}{\sqrt{\beta}}$

Three cases can be analyzed:

1. If $\sqrt{1+\delta} - \delta > 0$, the derivative is always negative, this ensures the uniqueness of the equilibrium
2. If $\sqrt{1+\delta} - \delta < 0$ there are two cases:
   (a) Either $-\frac{(\frac{1}{\sqrt{1+\delta}}+\frac{1}{1+\delta})}{(\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})}\frac{1}{\sqrt{\beta}} < \phi(0) = \frac{1}{\sqrt{2\pi}}$, in which case there exists a subset of values of z such that there are multiple solutions
   (b) Or $\frac{1}{\sqrt{\beta}(\frac{\delta-\sqrt{1+\delta}}{1+\sqrt{1+\delta}})} > \phi(0) = \frac{1}{\sqrt{2\pi}}$, in which case, we can ensure the uniqueness of the equilibrium again.

Finally there is a unique equilibrium iff $\sqrt{1+\delta}-\delta \geq 0$[11], or $\sqrt{1+\delta}-\delta < 0$ and $\sqrt{2\pi} > \sqrt{\beta}(\frac{\delta-\sqrt{1+\delta}}{1+\sqrt{1+\delta}})$, which can be summarized by the late inequality: $\sqrt{2\pi} > \sqrt{\beta}(\frac{\delta-\sqrt{1+\delta}}{1+\sqrt{1+\delta}})$, as we show in the following figure.

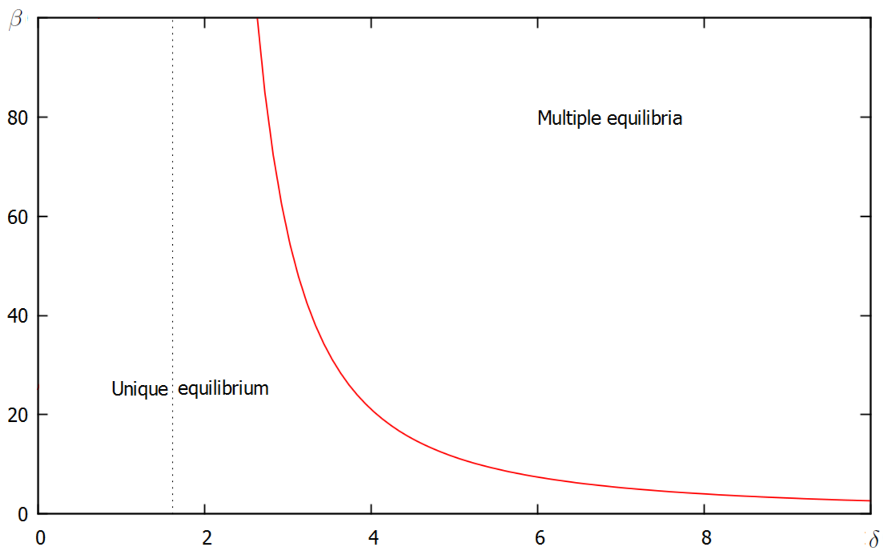

Figure 10: uniqueness vs. Multiplicity of equilibria: Conditions on the precisions

5.4 Analysis

This first model with an endogenous determination of the return factor has many interesting implications.

The first point we would like to highlight is that we find, in an intertemporal setting an effect of the amount of debt for refinancing analogous as the one found in HATTORI(2004). Indeed, with $T_1 = RN_0\Phi(s_0+\Phi^{-1}(\frac{1}{R})+\Phi^{-1}(\frac{1}{N_0}))$, the equilibrium value of $\psi = \frac{T_0*}{N_0}$ is given by $\psi = \frac{T_0*(R)}{N_0} = \Phi((\frac{1}{\sqrt{1+\delta}} - \frac{\delta}{1+\delta})\Phi^{-1}(\frac{1}{R}) - \frac{\delta}{1+\delta}\Phi^{-1}(\frac{1}{N_0}))$. We then have: $\frac{\partial\psi}{\partial N_0} = \frac{\delta}{1+\delta}\frac{1}{N_0^2}\frac{\phi((\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})\Phi^{-1}(\frac{1}{R})-\frac{\delta}{1+\delta}\Phi^{-1}(\frac{1}{N_0}))}{\phi(\Phi(\frac{1}{N_0}))} > 0$, which is consistent with the result found by HATTORI - which has been found in a non-intertemporal setting. This is, once again, perfectly consistent

[11] From $z = \Phi((\frac{1}{\sqrt{1+\delta}} - \frac{\delta}{1+\delta})\Phi^{-1}(\frac{1}{R}) - \frac{\delta}{1+\delta}\Phi^{-1}(\frac{1}{N_0})) + \frac{1}{\sqrt{\beta}}\left((\frac{1}{\sqrt{1+\delta}} + \frac{1}{1+\delta})\Phi^{-1}(\frac{1}{R}) + \frac{1}{1+\delta}\Phi^{-1}(\frac{1}{N_0})\right)$, it is clear that the case $\sqrt{1+\delta}-\delta = 0$ yields a unique associated $R(z)$.

with what we would have expected; the next-period incomes are approximately independent of the debt that must be paid back in the first period to avoid bankrupt. Hence, when this amount of debt increases -i.e. the refinancing pressure increases - the part of the past debt that can be reimbursed by newly issued bonds decreases for a given R. Hence, to avoid bankrupt, the part of the past debt that must be reimbursed by taxes will increase, and the fundamentals will need to be higher in order to avoid first-period default.

The second point we would like to talk about has not been pointed out yet. The sign of the expression $\sqrt{1+\delta}-\delta$, has two implications. The first one, that we just saw, enables to give a sufficient characterization for the uniqueness of the equilibrium. The second one, on which we will focus here, has more important economical implications: it provides the sign of $\frac{\partial T_0*}{\partial R}$. When $\sqrt{1+\delta}-\delta>0$, $\frac{\partial T_0*}{\partial R} = -\frac{N_0(\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})}{R^2}\frac{\phi((\frac{1}{\sqrt{1+\delta}}-\frac{\delta}{1+\delta})\Phi^{-1}(\frac{1}{R})-\frac{\delta}{1+\delta}\Phi^{-1}(\frac{1}{N_0}))}{\phi(\Phi(\frac{1}{R}))} < 0$. Let's focus on values of $T_0 \in [0; N_0]$. Provided $T_0 \in [0; N_0]$, the conditional probability of success is then: $p_0 = \frac{N_0 - T_0*}{N_0}$-still assuming a uniform distribution of T_0 on this range. Then, if $\sqrt{1+\delta}-\delta>0$, $\frac{\partial p_0}{\partial R}>0$. What does δ stand for? δ is the precision of the random variable s_0. This symbolizes in a way the degree of certainty that the agents have about the future -the second period- at the beginning of the first one, as opposed to β which would symbolize the degree of precision that the agents have about the present -the first period- at the beginning of the first one. This basically means that, if the agents are very uncertain about the future, the probability of success will rise when R increases. Conversely, if the agents are more certain about the future, $\sqrt{1+\delta}-\delta<0$, and $\frac{\partial p_0}{\partial R}<0$: an increase in R would lead to a decrease in the probability of success.

Once again, $\sqrt{1+\delta}-\delta$ captures the bargaining power of two opposite effects of R. The first one - $\sqrt{1+\delta}$ - as in the previous model, refers to a *direct payoff effect.* When R increases, the most direct effect is an increase in the expected payoff, for a given p: the individual that was the switching one, no longer is, as his expected payoff becomes positive, and there is an increase in the number of people who wish to buy the bonds.

The second one - δ - concerns the *informative effect* of R. Nonetheless the interpretation is quite different here. When R increases, it reduces the demand for the bonds. Indeed, we can consider the total amount of money that can be paid back the next period as fixed (for instance, the taxes that will be levied in the next period are independent of R and N_0). Thus, they know that an increase in R must be compensated by a decrease in the demand, for the State to be able to pay back all its creditors in the next period. Then, this informative effect can be interpreted as a *second-period default risk effect*: the agents react to an increase in R by a decrease in the demand in order to avoid an increase of the burden of the debt, which would lead to a second-period default. We will call it the *burden effect.*

When δ is sufficiently low, there is a lot of uncertainty concerning the future: in this case, the direct payoff effect exceeds the burden effect ($\sqrt{1+\delta}-\delta > 0$). Indeed, you already know very little about the future income: it then becomes very difficult to adjust your demand to a variation of R in the asset-supply market, because this market is too blurry for you. Hence, the stronger effect is the payoff effect, and you will buy more bonds if R increases: an increase in R would lead to a decrease in the first-period probability of default. You react in a myopic way.

On the other hand, when δ becomes higher, the burden effect will exceed the direct payoff effect. You have less uncertainty about the future, and you will then put more interest in ensuring the fact that the total repayment won't be too costly for the State in the next period. Hence, when R increases, you will decrease your demand in order to be sure to be reimbursed. You then react in a more forward looking way, and an increase in R would lead to an increase in the probability of default.

This more intertemporal model can also be adapted to the current situations in Europe. If you look at Greece for instance, there has been a lot of uncertainty about its future. Hence, the high increase in its returns has enabled to reduce the risk of default, by inducing the different agents to buy bonds, which increases the chances of reimbursing the past debt, and to avoid bankrupt. On the contrary, in Germany for instance, the uncertainty about the future is lower: hence, the lower the interest rate, the lower the risk of default. Indeed, as R decreases, the quantity of bonds that the State will be able to reimburse will be higher, and the State is sufficiently convincing concerning his ability for future reimbursement, to keep the agents increase their demand. Thus, this increases the chances of reimbursing the past debt too.

The same analysis as for the previous model can also be done: a country with a low R and an associated low risk of default is a country for which the agents have a great certainty about the future. When a crisis occurs, the agents suddenly fear the future, their uncertainty about the future increases, until the sign of $\sqrt{1+\delta}-\delta$ switches: in this case, to keep the risk of default lower, the value of R has to rise.

If this model has particularly interesting implications, this is annoying to notice that the introduction of the prior asset market gives birth to multiple equilibria for consistent values of β when δ becomes high. This model fails to try and discover a continuous link between the fundamentals and R. Indeed, having multiple equilibria necessarily imply at least one rupture in the continuity of R with respect to z, which induces that there will be at least some point where a slight change in z will lead to huge discrepancies in the realized return. Nonetheless, the assumptions put on the asset market game are very strong, in particular the assumption that the agents observe s_0 and T_0 in a melting way, and we are confident in the fact that a two-stage, micro-founded model that would erase the multiplicity of equilibria for a wider range of precisions of the signals could take place in future works, in the framework of global games analysis.

6 Concluding Remarks

In this paper, we tried and build a link between the literature of Global Games - that have proved since a long time its ability to remove the multiplicity of equilibria in diverse coordination games - and the literature on micro-founded debt models, where the multiplicity of equilibria has appeared in a great variety of works, from Calvo to Cole&Kehoe.

The main point of this paper has been to try and give a realistic signification to the value of the return on the government bonds. Until now, all the models that used Global Games theory to deal with sovereign debt issues gave little importance to the value of the interest rate; it was mainly set as exogenous, and used as a given variable, when it was time to solve the coordination games. Our belief is that, if R plays a central role in many micro-founded debt models, we should pay attention to this, while trying to transpose them into global games framework.

Our first attempt has been to consider R as given, in a *rational expectations equilibrium* analysis where the agents receive noisy signals about the fundamental of the economy. Nonetheless, we assumed that the value perceived by the agents (for instance secondary market players) reflected quite precisely the state of the economy, which gives an informational role to this return. This model enables, under classical restrictions, to solve for the uniqueness of equilibria, and gives a first framework to analyze the occurrence, and the probability of crisis. It also enables to give an informational reason to sovereign debt crises: if the agents don't trust anymore that R reflects the true value of the fundamentals, the better way for the State to avoid default is to let the value of R increase.

We then turned to a more complete model, where R became endogenously determined, and held the same kind of analysis, in a model which relies on the works of Hellwig&Al., that have been made on currency crises. Such a model has interesting implications in terms of sovereign debt crises. It enables to give an interesting analysis of such a crisis: the idea that this model conveys is that the evolution of the risk of default with the return heavily relies on the certainty about the future that the different agents have. If they are very uncertain about it, an increase in R would lead to a decrease in the probability of default. Otherwise, if you have precise information about the future, an increase in R would lead to an increase in the probability of default.

The problem is that as the vision that the agents have of the future becomes more precise, the conditions for the multiplicity to be removed induced a quite blurry signal about the fundamentals of the first period, which may seem unrealistic. Nonetheless it is important to keep in mind that the multiplicity of equilibria doesn't arise from the second-stage coordination game, but from the first-stage asset-pricing market; this means that to each value of R is associated a unique equilibrium, but the value of R may not be uniquely determined from a value of the fundamentals. Future research that would wish to solve for the uniqueness of equilibria in analogous models would have to focus on a better way to express the asset pricing game.

7 Appendix

So far, we have focused on the uniqueness of switching strategy equilibria. In fact, we will show that if there is a unique symmetric equilibrium in switching strategies, there can be no other equilibrium.

We will sketch the proof for the first, classical model, but the flavor of the proof is analogous for the two extensions of the model.

Let's denote by $u(T_0^i, \tilde{T_0})$ the expected utility from buying bonds when receiving the signal $T_0^i$when all other agents follow a switching strategy around $\tilde{T_0}$. As we saw, we have $u(T_0^i, \tilde{T_0}) = [p \times R - 1] \times N_0 = [\Phi(\sqrt{\beta}(T_0^i - T_0*))R - 1] \times N_0$.

We also still have $\Phi(\sqrt{\beta}(\tilde{T_0} - T_0*)) = \frac{T_0*}{N_0}$. This equality implicitly defines a unique $T_0 * (\tilde{T_0})$ given $\tilde{T_0}$.

We denote$b(\tilde{T_0})$ the unique value such that $u(b(\tilde{T_0}), \tilde{T_0}) = \Phi(\sqrt{\beta}(b(\tilde{T_0}) - T_0 * (\tilde{T_0})))R - 1 = 0$. This is the threshold value under which it is dominant not to buy bonds: any value T_0^i, such that $T_0^i < b(\tilde{T_0})$ implies that an agent won't buy bonds.

Hence, if i's opponent is following a switching strategy with cutoff $\tilde{T_0}$, i's best response is to follow a switching strategy with cutoff $b(\tilde{T_0})$.

One may notice that b(.) is an increasing function: indeed, $T_0 * (.)$is an increasing function, which makes $u(.,.)$ a function increasing in its first argument, and decreasing in its second argument. [12]

We also have, for all $\tau \in \overline{\mathbb{R}}$:

$$b(\tau) = \frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + T_0 * (\tau)$$

With $T_0 * (\tau) > 0$, this induces that $b(\tau) > \frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R})$.

The serie $(b^n(-\infty))_{n \in \mathbb{N}}$is then strictly increasing.

Similarly, for all $\tau \in \overline{\mathbb{R}}$:

$$b(\tau) = \frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + T_0 * (\tau)$$

With $T_0 * (\tau) < N_0$, this induces that $b(\tau) < \frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + N_0$.

The serie $(b^n(-\infty))_{n \in \mathbb{N}}$is both strictly increasing and bounded. Moreover, $b(.)$ admits a unique fixed point: $\frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + \frac{N_0}{R}$, which is the limit.

Similarly, $(b^n(+\infty))_{n \in \mathbb{N}}$is both strictly decreasing, and bounded.

Finally both series $b^n(-\infty)$ and $b^n(+\infty)$ tend to $\frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + \frac{N_0}{R}$ as $n \to \infty$.

[12] When x increases, $T_0 * (x)$ increases, and $u(., x)$ decreases: as $u(b(x), x)$, remains equal to zero and u(.,x) is increasing, this means we have to increase the first argument to keep $u(., x) = 0$

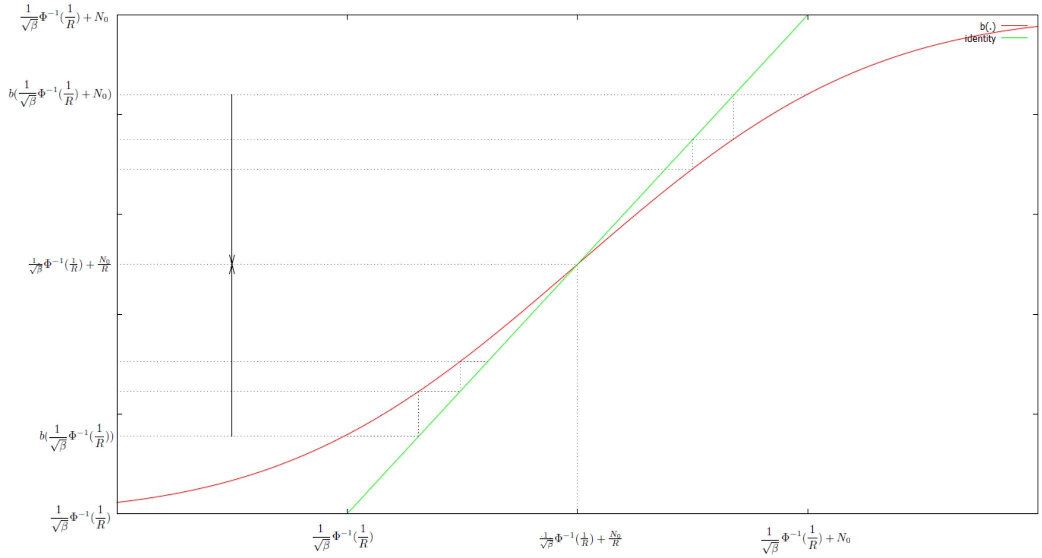

Figure 11: Graphical analysis of the series $(b^n(-\infty))_{n\in\mathbb{N}}$ and $(b^n(+\infty))_{n\in\mathbb{N}}$

The proof is intuitive when looking at the problem graphically.

We will now show by induction that, a strategy s surviving n rounds of iterated deletion of strictly dominated strategies is of the form:

$$s(T_0^i) = \begin{cases} 1 \; if \; T_0^i > b^n(\infty) \\ 0 \; if \; T_0^i < b^n(-\infty) \end{cases}$$

1. Case n=1

   Let's denote σ^{-i} the strategy profile of all players other than i. Let's denote $\tilde{u}(T_0^i, \sigma^{-i})$ the expected profit of i buying bonds, having observed signal T_0^i, when all other agents follow the strategy profile σ^{-i}. The probability of success is maximal when everyone buys bonds whatever the signal, and minimal when nobody buys bonds whatever the perceived signal.

   Hence, the expected profit of i buying bonds, having observed signal T_0^i, is such as:

$$u(T_0^i, \infty) \leq \tilde{u}(T_0^i, \sigma^{-i}) \leq u(T_0^i, -\infty)$$

From the monotonicity of u in its first argument, we get:

$$T_0^i < b(-\infty) \Rightarrow \tilde{u}(T_0^i, \sigma^{-i}) \leq u(T_0^i, -\infty) < u(b(-\infty), -\infty) = 0$$

And this is true whatever the strategy profile σ^{-i}. Hence, whatever the strategy profile of the other agents, if $T_0^i < b(-\infty)$, agent i will have an incentive not to buy bonds: $s(T_0^i) = 0$ when $T_0^i < b(-\infty)$.

Similarly,

$$T_0^i > b(\infty) \Rightarrow \tilde{u}(T_0^i, \sigma^{-i}) \geq u(T_0^i, +\infty) > u(b(+\infty), +\infty) = 0$$

Again, whatever the strategy profile of the other agents, if $T_0^i > b(+\infty)$, agent i will have an incentive to buy bonds: $s(T_0^i) = 1$ when $T_0^i > b(+\infty)$.

Any strategy profile $s(.)$ surviving the initial round of deletion of dominated strategies verifies:

$$s(T_0^i) = \begin{cases} 1 \; if \; T_0^i > b(\infty) \\ 0 \; if \; T_0^i < b(-\infty) \end{cases}$$

2. Induction

Suppose that the second proposition is true for an arbitrary n.

An agent i observes any signal T_0^i and knows that the other agents follow any strategy profile σ^{-i}, surviving n rounds of iterated deletions of dominated strategies.

Given this, the probability of default is maximal when σ^{-i} is the constant strategy profile of switching strategy with threshold $b^n(\infty)$, and minimal σ^{-i} is the constant strategy profile of switching strategy with threshold $b^n(-\infty)$.

We then have:

$$u(T_0^i, b^n(\infty)) \leq \tilde{u}(T_0^i, \sigma^{-i}) \leq u(T_0^i, b^n(-\infty))$$

The same arguments of monotonicity yield:

$$T_0^i < b^{n+1}(-\infty) \Rightarrow \tilde{u}(T_0^i, \sigma^{-i}) \leq u(T_0^i, b^n(-\infty)) < u(b(b^n(-\infty)), b^n(-\infty)) = 0$$

And:

$$T_0^i > b^{n+1}(\infty) \Rightarrow \tilde{u}(T_0^i, \sigma^{-i}) \geq u(T_0^i, +\infty) > u(b(b^n(\infty)), b^n(\infty)) = 0$$

This proves the property for the next stage.

We hence showed by induction that a strategy s surviving n rounds of iterated deletion of strictly dominated strategies is of the form:

$$s(T_0^i) = \begin{cases} 1 \; if \; T_0^i > b^n(\infty) \\ 0 \; if \; T_0^i < b^n(-\infty) \end{cases}$$

Recalling that $(b^n(-\infty))_{n\in\mathbb{N}}$ is strictly increasing, and that $(b^n(\infty))_{n\in\mathbb{N}}$ is strictly decreasing, any strategy that induces the agent to buy bonds for signals lower than $lim_{n\to\infty} b^n(-\infty) = \frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + \frac{N_0}{R}$, as any strategy that induces the agent not to buy bonds for signals higher than $lim_{n\to\infty} b^n(\infty) = \frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + \frac{N_0}{R}$ does not survive iterated dominance strategy. Moreover, there is an equilibrium in trigger strategies around $\frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + \frac{N_0}{R}$[13]. Finally, there is one, and only one strategy surviving the elimination of all the iteratively dominated strategies, and this strategy becomes the only equilibrium strategy, given that the rules of the global game (payoff functions, distribution function of T_0, ε_i ..) and the player's rationality are common knowledge.[14]

[13]Once again, due to monotonicity arguments. Indeed, $T_0^i < \frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + \frac{N_0}{R} \iff u(T_0^i, \frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + \frac{N_0}{R}) < u(\frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + \frac{N_0}{R}, \frac{1}{\sqrt{\beta}}\Phi^{-1}(\frac{1}{R}) + \frac{N_0}{R}) = 0$, and conversely

[14]Such assumptions may appear quite strong, as the original motivations of the introduction of these fuzzy signals - as stated in Carlsson&VanDamme(1993) - has been to try and relax the common knowledge assumption, but are widely used in this literature, and used here too.

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